Lag times and parameter mismatches in synchronization of unidirectionally coupled chaotic external cavity semiconductor lasers

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We report an analysis of synchronization between two unidirectionally coupled chaotic external cavity master and slave semiconductor lasers with two characteristic delay times, where the delay time in the coupling is different from the delay time in the coupled systems themselves. We demonstrate that parameter mismatches in photon decay rates for the master and slave lasers can explain the experimental observation that the lag time is equal to the coupling delay time.

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There are different types of synchronization in interacting chaotic systems. Complete, generalized, phase, lag, and anticipating synchronizations of chaotic oscillators have been described theoretically and observed experimentally. Complete synchronization implies coincidence of states of interacting systems, $y(t)=x(t)$ [1]. A generalized synchronization, introduced for drive-response systems, is defined as the presence of some functional relation between the states of response and drive, i.e., $y(t)=F(x(t))$ [2]. Phase synchronization means entrainment of phases of chaotic oscillators, $n\Phi_x - m\Phi_y = \text{const}$ (n and m are integers), whereas their amplitudes remain chaotic and uncorrelated [3]. Lag synchronization appears as a coincidence of shifted-in-time states of two systems, $y(t)=x_\tau(t)\equiv x(t-\tau)$ with positive τ and has been studied in between symmetrically coupled nonidentical oscillators [4]. Anticipating synchronization [5] also appears as a coincidence of shifted-in-time states of two coupled systems, but in this case the driven system anticipates the driver, $y(t)=x(t+\tau)$ or $x=y_\tau,\ \tau>0$. An experimental observation of anticipating synchronization has been reported recently [6].

Chaos synchronization is of fundamental importance in a variety of complex physical, chemical, and biological systems [7]. Because of their ability to generate high-dimensional chaos, time-delayed systems [8] are good candidates for secure communications based on chaos synchronization. In this context particular emphasis is given to the use of chaotic external cavity semiconductor lasers, because laser systems with optical feedback are prominent representatives of time-delayed systems that can generate hyperchaos [9].

Most experimental investigations of chaos synchronization in unidirectionally coupled external cavity semiconductor lasers [9] have found that the lag time between the master and slave lasers' intensities is equal to the coupling delay, whereas early numerical and analytical results [10,11] show that the lag time should be equal to the difference between the delay time in the coupling and round-trip time of the light in the transmitter's external cavity. Knowledge of the exact lag time is of considerable practical importance, as the recovery of message at the receiver critically depends on the correction made for the lag time [4,12].

Recently, there have been several attempts to explain the coupling-delay lag time synchronization in unidirectionally coupled external cavity semiconductor lasers. In Ref. [13] this phenomenon was related to a strong coupling and/or frequency detuning between the two lasers. However in a recent paper [14], where a *numerical* study of two unidirectionally coupled single-mode semiconductor lasers subject to optical feedback is reported, it was shown that such a phenomenon can be observed without any frequency detuning between the two lasers. In Ref. [14] it was found that two fundamentally different types of chaotic synchronization can occur depending on the relation between the strengths of the coupling and of the feedback of the lasers. In the first type of synchronization, when the feedback rates of the transmitter and receiver lasers are equal, the lag time is equal to the coupling delay between the transmitter and receiver lasers; in the second type of synchronization, when the feedback rate of the transmitter is equal to the sum of the feedback rate of the receiver and coupling strength, the lag time is the difference between the coupling delay and the round-trip time of the light in the transmitter. In numerical investigations of the first type of synchronization reported in Ref. [8] it was found that the synchronization error does not decay to zero but rather shows small oscillations even when the authors consider modified synchronization manifolds for the electric field amplitude and the carrier density by the introduction of a constant correction coefficient. Thus as the authors of Ref. [14] themselves acknowledge, the synchronization manifold introduced there is not perfect, but is approximate in nature.

In this paper we demonstrate that parameter mismatches in photon decay rates for the master and slave lasers can explain *perfect* synchronization with the lag time between the synchronized states that is equal to the coupling-delay time.

An appropriate framework for treating the evolution of the electric field of external cavity laser diodes is provided by the widely utilized Lang-Kobayashi equations [15],

$$\frac{dE_{1,2}}{dt} = \frac{(1+\imath\alpha_{1,2})}{2}\left(\frac{G_{1,2}(N_{1,2}-N_{01,02})}{1+s_{1,2}|E_{1,2}|^2} - \gamma_{1,2}\right)E_{1,2}(t) + k_{1,2}E_{1,2}(t-\tau_1)\exp(-\imath\omega\tau_1) + k_3E_1(t-\tau_2)\times\exp(-\imath\omega\tau_2), \tag{1}$$

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$$\frac{dN_{1,2}}{dt} = J_{1,2} - \gamma_{e1,e2}N_{1,2} - \frac{G_{1,2}(N_{1,2}-N_{01,02})}{1+s_{1,2}|E_{1,2}|^2}|E_{1,2}|^2,$$

where $E_{1,2}$ are the slowly varying complex fields for the master and slave lasers, respectively; $N_{1,2}$ are the carrier densities; $\gamma_{1,2}$ are the cavity losses; $\alpha_{1,2}$ are the linewidth enhancement factors; $G_{1,2}$ are the optical gains; $k_{1,2}$ are the feedback levels; k_3 is the coupling rate; ω is the optical frequency without feedback (no frequency detuning between the two lasers); τ_1 is the round-trip time in the external cavity; τ_2 is the time of flight between the master laser and the slave laser coupling-delay time; $J_{1,2}$ are the injection (pump) currents; $\gamma_{e1,e2}^{-1}$ are the carrier lifetimes; $s_{1,2}$ are the gain saturation coefficients. The term containing k_3 exists only for the slave laser, and accounts for the light injected from the master laser to the slave laser.

Now we shall demonstrate that depending on the laser systems' parameters, Eqs. (1) can allow for two regimes of lag synchronization between the lasers' intensities (which are related to the electric field amplitudes by $I \propto |E|^2$).

First we explore the possibility of perfect synchronization between the chaotic intensities of the master and slave lasers with the lag time equal to the coupling-delay time—as is found in most experimental cases,

$$I_{1,\tau_2} = I_2. \tag{2}$$

We also assume an analogous synchronization manifold for the carrier densities: $N_{1,\tau_2} = N_2$. As was numerically shown in Ref. [14] such a synchronization manifold (with some modifications) can exist if $k_1 = k_2$. However as mentioned above, the modified synchronization manifold studied in Ref. [14] was not perfect even after introducing a constant scaling factor $a = 1.016$ for the electric field amplitudes. In our notation the synchronization manifold considered in Ref. [14] would have been written as $a^2 I_{1,\tau_2} = I_2$. We also note the following further difference between the synchronization manifolds considered in this work and those in paper [14]: namely, above we assume the synchronization manifold $N_2 = N_{1,\tau_2}$, the analogous synchronization manifold in Ref. [14] is of the form $N_2 = N_{1,\tau_2} + \Delta_N$, where Δ_N is some constant. We would like to emphasize that even after such modifications of the synchronizations manifolds made in Ref. [14], synchronization was not perfect. The authors of Ref. [14] with reference to Mirasso (Ref. [13] in Ref. [14]) indicate that perfect synchronization is possible if different photon lifetimes are assumed for the master and slave lasers. A similar idea for achieving perfect synchronization between master and slave lasers with coupling-delay lag time is indicated in our recent work [16].

In this paper we show that perfect synchronization can be achieved without the above mentioned modifications of the synchronization manifolds. Following Ref. [16], suppose that there are parameter mismatches between the master and slave laser photon decay rates: $\gamma_1 \neq \gamma_2$. Using Eqs. (1) we write the dynamical equation for the E_{τ_2} in the following manner:

$$\frac{dE_{1,\tau_2}}{dt} = \frac{(1+\imath\alpha_1)}{2}\left(\frac{G_1(N_{1,\tau_2}-N_{01})}{1+s_1|E_{1,\tau_2}|^2} - \gamma_1\right)E_{1,\tau_2} + k_1 E_{1,\tau_1+\tau_2}\exp(-\imath\omega\tau_1).$$

Assuming that the laser parameters are identical (except for photon decay rates and feedback rates), we find that under the conditions

$$k_1 = k_2 \tag{3}$$

and

$$\frac{(1+\imath\alpha)}{2}\gamma_1 = \frac{(1+\imath\alpha)}{2}\gamma_2 - k_3\exp(-\imath\omega\tau_2), \tag{4}$$

(where $\alpha = \alpha_1 = \alpha_2$) the equations for E_{1,τ_2} and E_2 become identical and the lag synchronization manifold (2) exists. One can easily rewrite condition (4) in the more appealing form

$$(\gamma_2 - \gamma_1)^2 = \frac{4k_3^2}{1+\alpha^2}. \tag{5}$$

Thus in this paper we demonstrate analytically that by taking into account differences in the photon lifetimes for the master and slave lasers one can obtain perfect synchronization. Numerical simulations fully support the analytical approach. We perform simulations of Eqs. (1) by employing a Runge-Kutta-Fehlberg algorithm [17] for the following parameters: $\alpha_1 = \alpha_2 = 3$, $s_1 = s_2 = 0$, $N_{01} = N_{02} = 1.7\times 10^8$, $G_1 = G_2 = 2.14\times 10^4$, $\tau_1 = 10$ ns, $\tau_2 = 15$ ns, $\gamma_{e1} = \gamma_{e2} = 1$ ns^{-1}, $2\pi c/\omega = 635$ nm, $k_1 = k_2 = 10$ ns^{-1}, $k_3 = 30$ ns^{-1}, $\gamma_1 = \sqrt{3}.6\times 10^{10}$ s^{-1}, $\gamma_2 = 2\gamma_1$, the pump currents in units of the electron charge exceed the threshold value of the solitary laser $\gamma_{e1}N_{01}$ by factor 1.02. Notice that conditions (3) and (5) are satisfied. Figure 1 shows the time series of the (1) amplitude of the electric field $[E_{1,2} = A_{1,2}\exp(\imath\Phi_{1,2})]$ injected into the receiver $A_1(t-\tau_2)$; (2) amplitude of the electric field produced by the receiver $A_2(t)$; (3) synchronization error $\Delta A = |A_1(t-\tau_2) - A_2(t)|$. Perfect synchronization with the coupling-delay lag time is evident.

The necessary conditions for perfect synchronization (3) and (5) do not provide information on the stability of the synchronization manifold (2). Unfortunately it is practically very difficult to study analytically the stability of the synchronized solution. For that purpose one can use numerical simulations.

Figure 2 shows numerical simulations of Eqs. (1) for the feedback rates $k_1 = k_2 = 25$ ns^{-1} the other parameters are as in Fig. 1. One can notice that synchronized solution (2) is unstable. We have found that a perfect synchronized solution for $k_1 = k_2 = 25$ ns^{-1} can be made stable if k_3 exceeds 55 ns^{-1}, e.g., $k_3 = 60$ ns^{-1} for $\gamma_1 = 1.2\times 1/\sqrt{10}\times 10^{11}$ s^{-1}, $\gamma_2 = 2\gamma_1$. The general tendency is that with increasing feedback rate k_1 the coupling rate k_3 should be increased to maintain stability of the synchronized solution. We also notice that conditions (3) and (5) coincide with those derived in

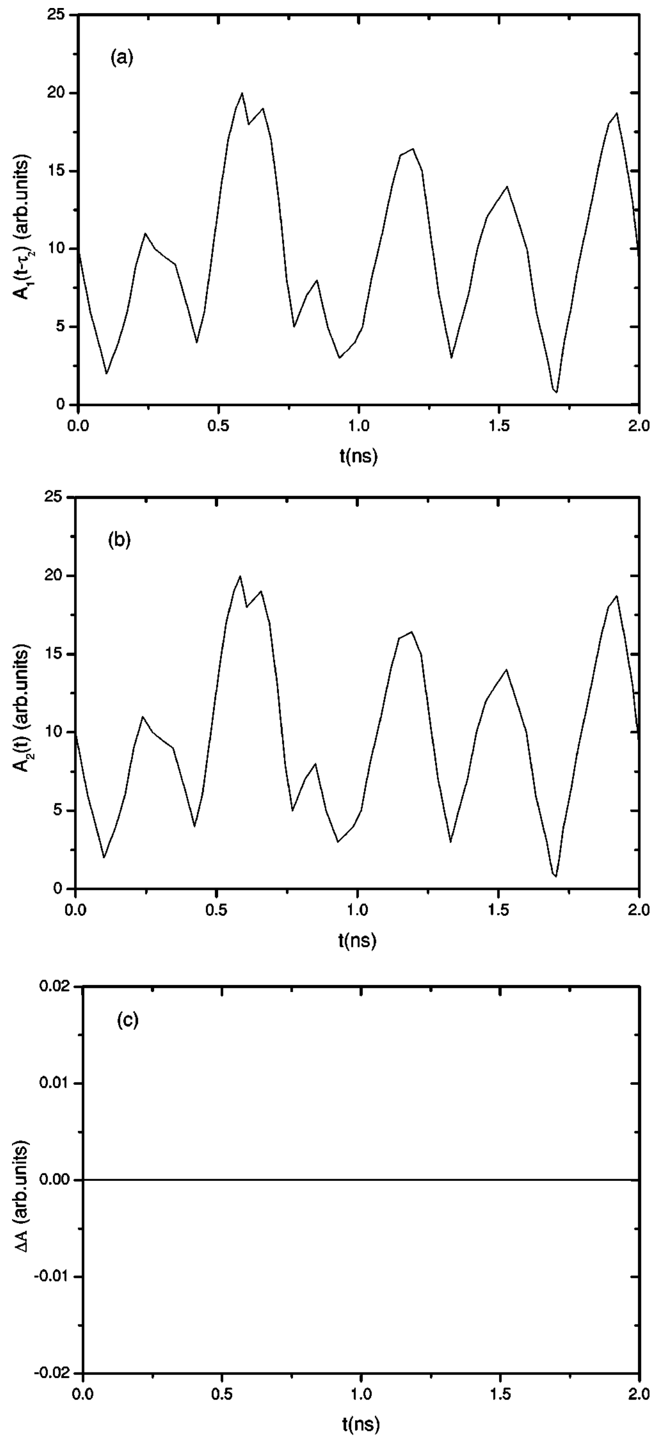

FIG. 1. Perfect lag synchronization with the coupling delay lag time: (a) the time series of the amplitude of the electric field injected into the receiver $A_1(t-\tau_2)$; (b) amplitude of the electric field produced by the receiver $A_2(t)$; (c) synchronization error $\Delta A = |A_1(t-\tau_2)-A_2(t)|$.

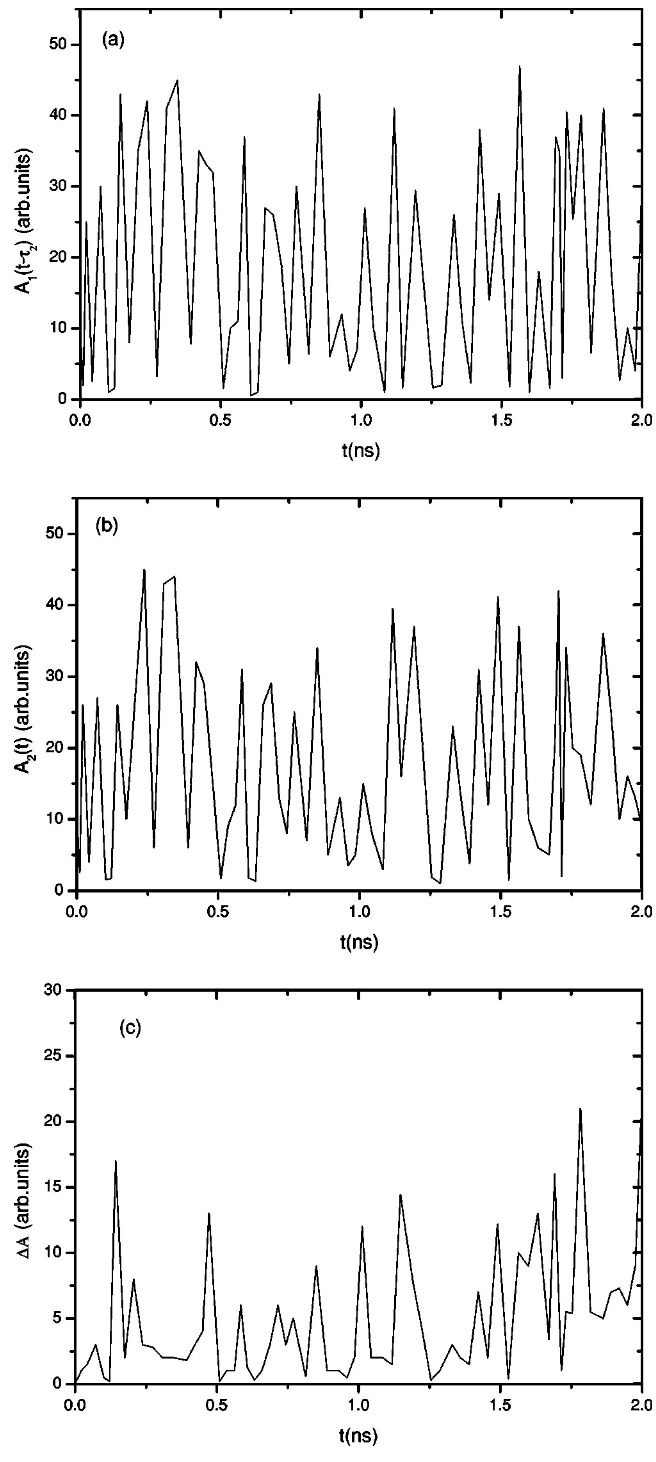

FIG. 2. Instability of the synchronized solution: (a) the time series of the amplitude of the electric field injected into the receiver $A_1(t-\tau_2)$; (b) amplitude of the electric field produced by the receiver $A_2(t)$; (c) synchronization error $\Delta A = |A_1(t-\tau_2)-A_2(t)|$.

Ref. [18] from the Lang-Kobayashi equations written for the intensity and the phase of the laser.

In real systems the condition for perfect synchronization cannot be fully satisfied. In consequence, it is of immense practical importance to consider the possibility of synchronization in the presence of deviation from the perfect synchronization condition. As noted in Ref. [19], since the coupling between the laser systems is unidirectional, one can set the coupling-delay time to zero which is justified by experiments [19] demonstrating that the observed synchronization phenomena are independent of τ_2. In particular, all the results would hold for $\tau_2 = 0$, i.e., for complete synchronization. As shown in the study of complete synchronization in Refs. [18,20], if the lasers are subject to the same feedback level and photon decay rates differ such that $\gamma_2 = \gamma_1 + \delta$ with δ being an arbitrary type of synchronized solutions $I_2 = bI_{1,\tau_2}, N_2 = N_{1,\tau_2} + \Delta_N$, where b and Δ_N are some constants, can be obtained. Such a functional relation between the states of the master and slave laser systems corresponds to generalized synchronization of the coupled systems [2]. Thus in the presence of a deviation from the exact (perfect) synchronization conditions, generalized synchronization between the unidirectionally coupled master and slave laser systems can be obtained with a lag time τ_2. Finally we underline that in the absence of parameter mismatches when the feedback rate of the transmitter is equal to the sum of the feedback rate of the receiver and coupling strength $k_1 = k_2 + k_3$, the lag time is the difference between the coupling delay time τ_2 and the round-trip time of the light in the transmitter τ_1, in full accordance with Refs. [10,11,14,18].

To summarize, we have studied synchronization between unidirectionally coupled chaotic external cavity semiconductor lasers with two characteristic delay times, where the delay time in the coupling is different from the delay time in the coupled systems themselves. We have demonstrated that parameter mismatches in photon decay rates for the master and slave lasers can explain the experimental observation that the lag time is equal to the coupling delay time and derived relevant existence conditions. The concept of lag synchronization was introduced by Rosenblum *et al.* [4] under certain approximations in studying synchronization between *bidirectionally* coupled systems described by ordinary differential equations (no intrinsic delay terms) with *parameter mismatches* (for recent progress in the investigation of lag synchronization, see also Ref. [21]). As such, lag synchronization cannot be observed if two oscillators are completely identical [22]. In this paper we have demonstrated that the presence of parameter mismatches is also essential for the existence of *perfect* retarded synchronization with the coupling-delay lag time between certain classes of *unidirectionally* coupled time-delayed systems. To be more specific, for the example of external cavity laser diodes we have shown that for unidirectionally coupled time-delayed systems *perfect* retarded synchronization occurs both with and without parameter mismatches; parameter mismatches can change the lag time between the synchronized systems.

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